Quantum sensors: atomic interferometry

Quantum technologies and industry

Kenneth MAUSSANG

Université de Montpellier

Ecole Centrale Casablanca

2020 – 2021

Quantum sensors: atomic interferometry

I.1. Atomic structure

Without spin-orbit interaction, eigenstates of hydrogen-like atoms can be expressed in basis of mutually commuting operators: $\hat{H}_0$, $\hat{\mathbf{L}}^2$, $\hat{L}_z^2$, $\hat{\mathbf{S}}^2$ and $\hat{S}_z^2$, where $\hat{H}_0$ is the hamiltonian associated to the electron motion (spatial degrees of freedom). In this case, electrons

are fully described with four quantum number n, l, m_l and m_s. The

principal quantum number n is associated to the electron energy, typically in the eV range (**optical transitions**).

Within a given n value, without spin-orbit coupling, all states are energy degenerated. The azimuthal quantum number l is associated to the eigenvectors of $\hat{\mathbf{L}}^2$ such that

$$\hat{\mathbf{L}}^2 \left| n, l, m_l, m_s \right\rangle = \hbar^2 l(l+1) \left| n, l, m_l, m_s \right\rangle, \quad l \in [\![0, n-1]\!].$$

In chemistry and spectroscopy, $l = 0$ is called an s orbital, $l = 1$ a p orbital, $l = 2$ a d orbital, and $l = 3$ an f orbital. The projection of the orbital momentum along z axis provides m_l quantum number, such that

$$\hat{L}_z \left| n, l, m_l, m_s \right\rangle = \hbar m_l \left| n, l, m_l, m_s \right\rangle, \quad l \in [\![-l, l]\!].$$

Similarly, m_s is the spin quantum number, taking into account that an electron is a spin half particle, $S = 1/2$ and $m_S \in \{-1/2, +1/2\}$.

In atoms, the fine structure originates from the coupling between $\hat{\mathbf{L}}$ is the angular momentum of the electron and $\hat{\mathbf{S}}$ the spin of the electrons. For a general potential of interaction with the nucleus $V(r)$, it is possible to show that this coupling is described by the following hamiltonian

$$\hat{H}_{\mathrm{f}} = \frac{1}{2m^2c^2}\frac{1}{r}\left(\frac{\partial V}{\partial r}\right)\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}.$$

For a hydrogen-like atom,

$$V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r},$$

so that the spin-orbit coupling hamiltonian as the following expression

$$\hat{H}_{\mathrm{f}} = \frac{1}{2m^2c^2}\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r^3}\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}.$$

With spin-orbit, total Hamiltonian no longer commutes with $\hat{L}_z$ or $\hat{S}_z$. One needs to exploit degeneracy of $\hat{H}_0$ and found a new basis in which $\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ is diagonal. In that goal, one introduces the total orbital momentum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$. It is straightforward that

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{S}} = \frac{1}{2}\left(\hat{\mathbf{J}}^2 - \hat{\mathbf{L}}^2 - \hat{\mathbf{S}}^2\right).$$

Combining a spin 1/2 with angular momentum l, the total angular momentum $\hat{J}$ can take values

$$J = l \pm 1/2, \text{ and } m_J \in [\![-J, J]\!].$$

For a hydrogen-like atom, the energy shift induced by spin-orbit coupling depends on n and J as follow

$$\Delta E_{n,J=1\pm l,m_J,l,m_S} = \frac{1}{2}mc^2\left(\frac{\alpha E}{n}\right)^4\left(\frac{3}{4}-\frac{n}{J+1/2}\right),$$

where

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c},$$

is the *fine structure constant*. The fine structure of energy levels of atoms is also corrected for some relativistic effects such as the Lamb shift. Fine structure results in level splitting of the gross initial structure with energy shift in the 10^{-5} to 10^{-4} eV range.

I.1. Atomic structure

In spectroscopy, for a state with principal quantum number n, total spin S, orbital angular momentum l and total angular momentum J, one may define the state by the spectroscopic notation

$$n^{2S+1}L_J,$$

with $L \in \{S, P, D, F\}$. For hydrogen-like atom, with a single electron, $2S + 1 = 2$. In this case, the factor $2S + 1$ is just dropped for brevity. Example: $2P_{3/2}$ level.

The atomic hyperfine structure results from the interaction between the nuclear spin $\hat{\mathbf{I}}$ and the total angular momentum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$, where $\hat{\mathbf{L}}$ is the angular momentum and $\hat{\mathbf{S}}$ the spin.

The appropriate quantum observable is $\hat{\mathbf{F}} = \hat{\mathbf{I}} + \hat{\mathbf{J}}$, the total orbital momentum. The hamiltonian associated to the nuclear spin - orbital spin coupling is commonly written as

$$\hat{H}_{\text{hf}} = A\hat{\mathbf{I}} \cdot \hat{\mathbf{J}}.$$

The operator $\hat{\mathbf{I}} \cdot \hat{\mathbf{J}}$ might be rewritten as follow

$$\hat{\mathbf{I}} \cdot \hat{\mathbf{J}} = \frac{1}{2}\left(\hat{\mathbf{F}}^2 - \hat{\mathbf{I}}^2 - \hat{\mathbf{J}}^2\right)$$

In the appropriate basis (eigenvectors of $\hat{F}$, $\hat{J}$ and $\hat{I}$), the energy shift due to nuclear spin - electronic orbital momentum coupling is

$$\Delta E_{F,J,I} = \frac{\hbar^2}{2} A\left(F(F+1) - I(I+1) - J(J+1)\right),$$

with $F \in [\![|J-I|, J+I]\!]$. Hyperfine structure results in level splitting of the initial fine structure with energy shift in the 10^{-6} to 10^{-5} eV range.

Hyperfine structure results in level splitting of the initial fine structure with energy shift in the 10^{-6} to 10^{-5} eV range.

In spectroscopy, levels are design with both the fine structure notation but adding the information on the F quantum number, for example the ${}^2S_{1/2}$, $F = 0$ state.

I.1. Alkali atoms

An isolated atom is a very sensitive system to external perturbations.

Let's consider the case of ^{133}Cs atom. It's an alkali atom, well-known in metrology (the International System of Units (SI) has defined the second as the duration of 9,192,631,770 cycles of radiation corresponding to the transition between two hyperfine levels of the ground state of the ^{133}Cs atom).

Beyond Cs, alkali atoms have energy level structures well-adapted for laser cooling, and commonly used in ultracold atoms experiments or for production of atomic Bose-Einstein condensates. In such atoms, the hyperfine splitting of the ground state typically lies in the GHz range

Atom	D_1 line	wavelength
Cs	$6^2S_{1/2} \longrightarrow 6^2P_{1/2}$	$894.592,959,86(10)$ nm
Na	$3^2S_{1/2} \longrightarrow 3^2P_{1/2}$	$589.755,814,7(15)$ nm
^{87}Rb	$5^2S_{1/2} \longrightarrow 5^2P_{1/2}$	$794.978,851,156(23)$ nm
^{85}Rb	$5^2S_{1/2} \longrightarrow 5^2P_{1/2}$	$794.979,014,933(96)$ nm

Atom	D_2 line	wavelength
Cs	$6^2S_{1/2} \longrightarrow 6^2P_{3/2}$	$852.347,275,82(27)$ nm
Na	$3^2S_{1/2} \longrightarrow 3^2P_{3/2}$	$589.158,326,4(15)$ nm
^{87}Rb	$5^2S_{1/2} \longrightarrow 5^2P_{3/2}$	$780.241,209,686(13)$ nm
^{85}Rb	$5^2S_{1/2} \longrightarrow 5^2P_{3/2}$	$780.241,368,271(27)$ nm

Atom	Hyperfine splitting	frequency
Cs	$6^2S_{1/2},\ F=3 \longrightarrow F=4$	$9.192,631,770$ GHz (exact)
Na	$3^2S_{1/2},\ F=1 \longrightarrow F=2$	$1.771,626,128,8(10)$ GHz
^{87}Rb	$5^2S_{1/2},\ F=1 \longrightarrow F=2$	$6.834,682,610,904,290(90)$ GHz
^{85}Rb	$5^2S_{1/2},\ F=1 \longrightarrow F=2$	$3.035,732,439,0(60)$ GHz

In an alkali atom, the fundamental state is split into **two hyperfine states, and might be seen as a two-level system**. With a hyperfine splitting in the GHz range, the lifetime is rather long and spontaneous emission can be neglected.

$$\Gamma_{\text{rad}}(\omega) = \frac{\omega^3 n \left| \langle 0 | \hat{\mathbf{d}} | 1 \rangle \right|^2}{3\pi\varepsilon_0 \hbar c^3},$$

I.1. Atoms as sensors

The energy level structure of a atom might be affected both by a magnetic field or an electrical field. A field changes transitions frequency.

Measuring a frequency shift of a transition permit to calculate the corresponding field.

Recently, atomic magnetometer with microfabricated cell of Cs vapor has been reported, with sensitivity below 100 $fT/\sqrt{Hz}$ range, a bandwidth close to 1 kHz for scalar field measurement below the pT range. Moreover, this experimental setup is suited for portable devices.

Another research team has reported 0.54 $fT/\sqrt{Hz}$ sensitivity but with a lab setup. Such a magnetometer based on laser measurements of atomic energy levels can detect a magnetic field one hundred billion times smaller than the Earth's.

I.1. Atoms as sensors

Atoms are massive particles, and can be used to measure gravity. Atomic gravimeters are based on free fall of an atomic cloud combined to an atomic interferometry scheme. Gravimeter have many applications such as oil prospection.

Rotations might be measured also thanks to atomic interferometer, based on Sagnac's effect, and are used as precision gyrometers.

Muquans is a supplier of integrated quantum solutions, more specifically absolute atomic gravimeters and atomic clocks (`https://www.muquans.com`). This company has developed quantum inertial sensors (gravimeters), as a spinoff company from research labs. The absolute gravimeter of Muquans reach the 50 μGal/$\sqrt{\text{Hz}}$ range[1] in a quiet place, with 2 Hz measurement frequency and a long-term stability better than 1 μGal.

[1] 1 μGal$= 10^{-8}$ m·s^{-2}.

Absolute atomic gravimeter proposed by Muquans (https://www.muquans.com).

I.1. Atoms as sensors

Atoms are used for time measurement, as a frequency etalon. The hyperfine transition $F = 3 \longrightarrow F = 4$ of the $6^2S_{1/2}$ ground state of Cs is fixed at $9.192, 631, 770$ GHz as a consequence of the SI unit definition of the second.

The technical challenge for time measurement consists in measurement this transition frequency without energy shift due to external electrical or magnetic field, or due to interactions between atoms in the cloud probed.

Time metrology is an important research activity of atomic physics. Precise and transportable clocks are of importance for geopositioning systems such as GPS or Galileo.

Let consider an atom, modeled by a two level system $\{|0\rangle, |1\rangle\}$. It is equivalent to an effective spin, an is well-suited to Bloch sphere formalism. These two levels might be the two hyperfine states of the ground state of the atom and might be manipulation with microwaves radiation. Without radiation, in the $\{|0\rangle, |1\rangle\}$ basis, the Hamiltonian of the atom is

$$\hat{H}_0 = \hbar\omega_0 \, |1\rangle\langle 1| = \begin{pmatrix} \hbar\omega_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Each pure state can be written in the following form

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|1\rangle.$$

The corresponding point on the Bloch's sphere is the point on the unit sphere in $\mathbb{R}^3$ which has the same polar angles (θ, φ). Free evolution of an atom is the following

$$|\psi(t)\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi - i\omega_0 t}|1\rangle,$$

which is geometrically interpreted on the Bloch's sphere as a rotation at ω_0 along the axis Oz.

Now one considers that the atom interacts with a radiation at ω, $E(t) = E_0 \cos(\omega t + \phi)$. The interaction Hamiltonian $\hat{H}_I$ is a dipolar coupling such that

$$\hat{H}_I = \hat{d}E(t) = \frac{\hbar\Omega}{2}\cos(\omega t + \phi)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with

$$\Omega = \frac{2dE_0}{\hbar}, \quad d = \langle 1|\hat{d}\,|0\rangle.$$

Ω is the Rabi frequency, quantifying the coupling between the atom and the radiation.

Now let's move to the rotating frame at ω along Oz, applying the following unitary operation

$$\hat{U}(t) = \exp\left(i\omega t \left|1\right\rangle\left\langle 1\right|\right).$$

In the interaction representation, $\left|\psi\right\rangle_{\text{int}} = \hat{U}(t)\left|\psi\right\rangle$, the interaction Hamiltonian of the light-atom coupling is then

$$\hat{H}_{I,\text{int}} = \hat{U}(t)\hat{H}_I\hat{U}^\dagger(t) = \frac{\hbar\Omega}{2}\begin{pmatrix} e^{i\phi} + e^{i(2\omega t+\phi)} & 0 \\ 0 & e^{-i\phi} + e^{-i(2\omega t+\phi)} \end{pmatrix}.$$

In the limit $\Omega, \delta \ll \omega_0$, with $\delta = \omega_0 - \omega$, one may use the widely used rotating wave approximation (RWA) in which one will neglect the terms $e^{\pm i(2\omega t+\phi)}$.

In this approximation, the full Hamiltonian in the rotating frame becomes

$$\hat{H} = \hbar\delta \left|1\right\rangle\left\langle 1\right| + \frac{\hbar\Omega}{2}\left(e^{i\phi}\left|1\right\rangle\left\langle 0\right| + e^{-i\phi}\left|0\right\rangle\left\langle 1\right|\right) = \hbar\begin{pmatrix} \delta & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}.$$

This is an effective spin-1/2 Hamiltonian. Adding a total energy of $-\hbar\delta/2$, it can be rewritten using the Pauli matrices $\hat{\vec{\sigma}} = \hat{\sigma}_x\vec{u}_x + \hat{\sigma}_y\vec{u}_y + \hat{\sigma}_z\vec{u}_z$

$$\hat{H} = \frac{\hbar}{2}\vec{\Omega}\cdot\hat{\vec{\sigma}}, \quad \text{with} \quad \vec{\Omega} = \begin{pmatrix} \Omega\cos\phi \\ \Omega\sin\phi \\ \delta \end{pmatrix}.$$

For a radiation of constant amplitude, it corresponds to a rotation of the initial state $|\psi(0)\rangle$ at the angular rotation vector $\vec{\Omega}$

$$|\psi(t)\rangle = \exp\left(-i\vec{\Omega}\cdot\hat{\vec{\sigma}}t/2\right)|\psi(0)\rangle.$$

In the case of a state initially in the ground state, $|\psi(0)\rangle = |0\rangle$, and a phase $\phi = 0$, this rotation results in so-called **Rabi oscillation**: the probability $P_{0\to 1}(t)$ to detect the atom in the state $|1\rangle$ after an interaction time t is

$$P_{0\to 1}(t) = \frac{\Omega^2}{\delta^2+\Omega^2}\sin^2\left(\frac{\sqrt{\delta^2+\Omega^2}}{2}t\right).$$

At resonance, the transition probability is simplified to

$$P_{0\to 1}(t) = \sin^2\left(\frac{\Omega t}{2}\right),$$

with maximal amplitude of oscillation.

I.2. Rabi oscillations and state preparation

This permits to manipulate the state of an atom and prepare it is a given state. For instance, going from $|0\rangle$ to $|1\rangle$, one just has to adjust the interaction time t_π with the radiation so that

$$\frac{\Omega t_\pi}{2} = \frac{\pi}{2} \Leftrightarrow t_\pi = \frac{\pi}{\Omega}.$$

This operation is called a π pulse, corresponding to a rotation on the Bloch sphere of an angle π. For an intermediate time between 0 and t_π, one obtains a superposition of states. In the particular case of $t_{\pi/2} = \frac{t_\pi}{2} = \frac{\pi}{2\Omega}$, a so-called $\frac{\pi}{2}$ pulse, providing the following state

$$|0\rangle \longrightarrow \frac{|0\rangle + i\,|1\rangle}{\sqrt{2}}.$$

I.2. Ramsey interrogation

A Ramsey interrogation of a two level system consists in the following sequence

- **Initialization** Atoms are prepared in the ground state $|0\rangle$.
- $\pi/2$ **pulse** Preparation of atoms in a superposition of states, on the equator of the Bloch's sphere.
- **Free evolution during a time** T Atoms evolves free without any radiation, and rotates on the equator at Larmor frequency.
- $\pi/2$ **pulse** State is rotated on the Bloch sphere.
- **Measurement** of the $\hat{\sigma_z}$ observable.

Assuming $\phi = 0$, the action of the Ramsey interrogation can be expressed by the following operator

$$\hat{R} = \exp\left(-i\frac{\pi}{2}\frac{\hat{\sigma}_x}{2} - i\delta t_{\pi/2}\right)\exp\left(i\varphi\frac{\hat{\sigma}_z}{2}\right)\exp\left(-i\frac{\pi}{2}\frac{\hat{\sigma}_x}{2} - i\delta t_{\pi/2}\right),$$

where $t_{\pi/2}$ is the $\pi/2$ pulse duration, and $\varphi = -\delta T$ with T the duration of interrogation. Usually, such sequence is realized near resonance such that $|\delta| \ll \Omega$ then

$$\hat{R} \approx \exp\left(-i\frac{\pi}{2}\frac{\hat{\sigma}_x}{2}\right)\exp\left(i\varphi\frac{\hat{\sigma}_z}{2}\right)\exp\left(-i\frac{\pi}{2}\frac{\hat{\sigma}_x}{2}\right).$$

The first $\pi/2$ pulse places atoms in the superposition of states

$$|0\rangle \xrightarrow{\pi/2 \text{ pulse}} \frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right).$$

During the period of free evolution T, the system rotates around the z-axis with the angular velocity δ. The superposition thus picks up a phase $\varphi = -\delta T$, ending up in the state

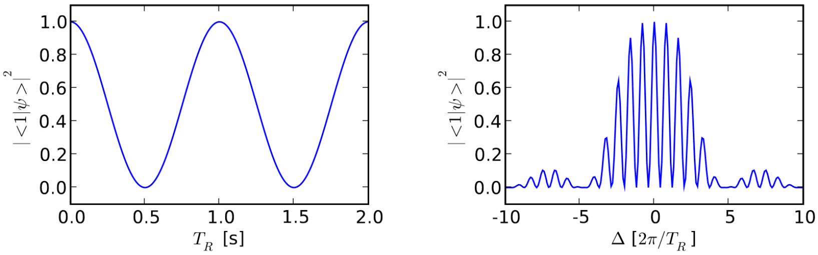
$$\frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right) \xrightarrow{\text{free evolution during } T} \frac{1}{\sqrt{2}}\left(|0\rangle + ie^{i\varphi}\,|1\rangle\right).$$

The second $\pi/2$ pulse rotates the atoms by $\pi/2$ around the x-axis on the Bloch's sphere. The resulting final state is then

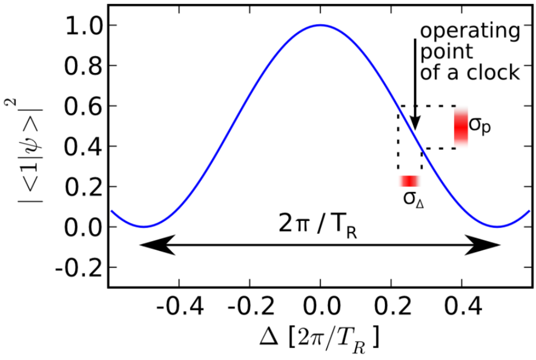
$$\frac{1}{\sqrt{2}}\left(|0\rangle + ie^{i\varphi}\,|1\rangle\right) \xrightarrow{\pi/2 \text{ pulse}} -\sin\left(\frac{\varphi}{2}\right)|0\rangle + \cos\left(\frac{\varphi}{2}\right)|1\rangle.$$

I.2. Ramsey interrogation

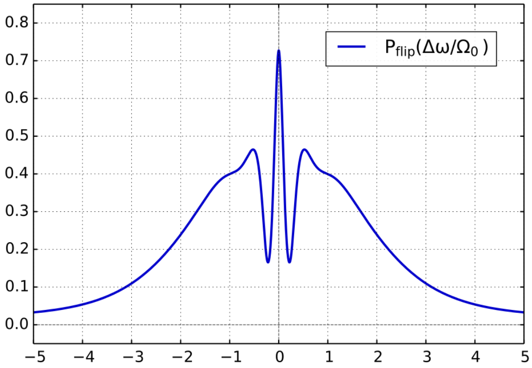
The measurable result of a Ramsey sequence is an excitation probability of the excited atomic state *i.e.* , experimentally, a population imbalance of the atomic state states. This imbalance is an oscillating function of $\varphi = -\delta T$. For a fixed detuning δ, it is a measure of T. For a fixed interrogation time T, it is a measurement of the detuning δ. Experimentally, either version can be realized and the result is refered to as Ramsey fringes in the time and frequency domain, respectively. In an atomic clock, T is always kept fixed, so that Ramsey interrogation yields a measurement of the detuning δ. To maximize the sensitivity to δ, the clock is operated on the slope of the central Ramsey fringe.



Ramsey fringes in the time domain (left, for $\delta = 2\pi \times 1$ Hz) and the frequency domain (right).

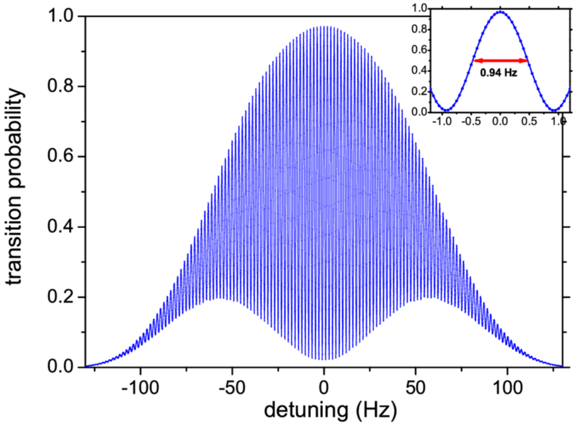


The central frequency–domain Ramsey fringe. Clocks are operated at the point of highest slope, where the transisition probability is the most sensitive to δ. Here, a given error σ_p on the measurement of the transition probability translates into a minimal error σ_δ on the frequency measurement.



Ramsey fringes with a thermal cloud.

`https://commons.wikimedia.org/wiki/File:Mplwp_ramsey_fringes_thermal.svg`



Ramsey fringes with ultracold atoms (atomic fountain). *Cold atom Clocks and Applications* `https://arxiv.org/abs/physics/0502117`

The maximum slope that can be obtained is

$$\left.\frac{\mathrm{d}p}{\mathrm{d}\delta}\right|_{\text{max}} = \frac{T}{2} = \frac{\pi}{2}\frac{Q_{\text{at}}}{\omega_0},$$

where Q_{at} is the quality factor of the clock,

$$Q_{\text{at}} = \frac{\omega_0 T}{\pi} = \frac{\omega_0}{\Delta},$$

with $\Delta = \pi/T$ the half width of the central Ramsey fringe.

Squeezing of a quantum state consists in the redistribution of quantum fluctuations between two observables that do not commute, while minimizing Heisenberg's uncertainty relationship. The quantum fluctuations of the two observables $\hat{A}$ and $\hat{B}$ verify Heisenberg's inequality according to

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|.$$

In the case of a spin, or an angular momentum, one obtains

$$\Delta\hat{J}_i\Delta\hat{J}_j \geq \frac{1}{2}\left|\left\langle\hat{J}_k\right\rangle\right|,$$

where $(i,j,k) \in \{x,y,z\}$, as well as all circular permutations.

The coherent spin state (denoted CSS) is defined as the eigenvector of the spin component in the direction (θ, φ),

$$\hat{J}_{\theta,\varphi} = \hat{J}_x \sin\theta\cos\varphi + \hat{J}_y \sin\theta\sin\varphi + \hat{J}_z \cos\theta,$$

with eigenvalue[2] J, where θ and φ are the polar and azimuthal angles. The CSS state $|\theta, \varphi\rangle$ minimizes the Heisenberg relation with standard deviations $\sqrt{\frac{J}{2}}$ equally distributed over any components orthogonal to the (θ, φ) direction.

[2]for a system containing N spin particles 1/2, $J = N/2$.

II. Spin squeezing

The squeezed spin state (denoted SSS) is the state for which the variance of one spin component orthogonal to the mean direction is smaller than the standard quantum limit, i.e. $\frac{J}{2}$.

If we consider a system of $2J$ spins, with $J = N/2$, and if all the spins initially point in the x direction, *i.e.* $\langle \hat{J}_x \rangle = N/2$, so $\Delta \hat{J}_z^2 = \left\langle \hat{J}_z^2 \right\rangle - \left\langle \hat{J}_z \right\rangle^2 = \left\langle \hat{J}_z^2 \right\rangle$. Moreover,

$$\left\langle \hat{J}_z^2 \right\rangle = \sum_{j,l} \langle \hat{s}_{z,j} \hat{s}_{z,l} \rangle = \sum_{j} \left\langle \hat{s}_{z,j}^2 \right\rangle + \sum_{j \neq l} \langle \hat{s}_{z,j} \hat{s}_{s,l} \rangle ,$$

and if there are no correlations between the individual spins, the variance of J_z is simply the sum of the individual variances, *i.e.*

$$\Delta \hat{J}_z^2 = \sum_{j} \left\langle \hat{s}_{z,j}^2 \right\rangle = \frac{N}{4}.$$

It is the case of a CSS state, with

$$\Delta \hat{J}_z = \Delta \hat{J}_y = \sqrt{\frac{\left\langle \hat{J}_x \right\rangle}{2}}.$$

These variances, from a system of uncorrelated N spins, then have a characteristic value called **quantum standard limit**. This is the "reference" variance for a given number of particles, the variance of the reference state that is the coherent state, made of N uncorrelated spins. To introduce correlations between the spins, a non-linear interaction must be used.

II. How to squeeze?

If we consider the simplest non-linear interaction, the Hamiltonian can be put in the following form

$$\hat{H} = \hbar\chi J_z^2.$$

Such hamiltonian is called "Kerr hamiltonian". Let consider a CSS as initial state $\left|\frac{\pi}{2}, 0\right\rangle$ in the direction $\theta = \frac{\pi}{2}$ and $\varphi = 0$ on the Bloch sphere. This state might be decomposed on the $|J, J-k\rangle$ basis such that

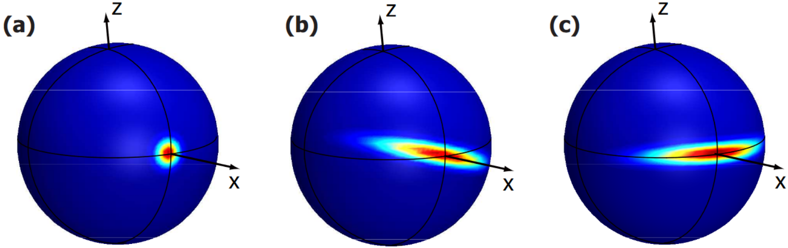
$$\left|\frac{\pi}{2}, 0\right\rangle = \frac{1}{2^J}\sum_{k=0}^{2J}\sqrt{\begin{pmatrix} 2J \\ k \end{pmatrix}}\,|J, J-k\rangle\,,$$

and $\left\langle \hat{\mathbf{J}} \right\rangle \simeq J\mathbf{u}_x$. The system will evolve according to a unitary transformation of evolution operator

$$\hat{U}(t) = \exp\left(-i\chi t\hat{J}_z^2\right).$$

As a result of this non-linear evolution, we still have $\left\langle \hat{\mathbf{J}} \right\rangle \propto \mathbf{u}_x$, but the coherent state has been "distorted" to give an ellipse on the Bloch sphere, of squeezed axis $\mathbf{u}_\perp$ for times shorter than $1/(|\chi|\sqrt{2J})$. This axis is then said *squeezed* in the sense that the variance of the operator $\hat{J}_\perp = \hat{\mathbf{J}} \cdot \mathbf{u}_\perp$ associated is below the standard quantum limit $J/2$. The conjugate axis has a variance greater than the standard quantum limit in order to satisfy the Heisenberg inequality. It is possible to transfer the squeezed property of the $\mathbf{u}_\perp$ component to any component perpendicular to $\mathbf{u}_x$.

II. How to squeeze?



To do this, a pulse is applied to generate a rotation of the state around the x axis, and align the compressed quasiprobability with this axis. The final state can be written as

$$|\psi(t)\rangle = \exp(-i\nu \hat{J}_x)\exp(-i\chi t \hat{J}_z^2)\left|\frac{\pi}{2}, 0\right\rangle . \qquad (1)$$

From the equation (1), one obtains the mean values and standard deviations of the different components of spin

$$\begin{aligned}
\left\langle \hat{J}_x \right\rangle &= J\cos^{2J-1}(\chi t),\ \left\langle \hat{J}_y \right\rangle = 0,\ \left\langle \hat{J}_z \right\rangle = 0,\\
\left\langle \Delta \hat{J}_x^2 \right\rangle &= \frac{J}{2}\left(2J\left(1-\cos^{2(2J-1)}(\chi t)\right) - \left(J-\frac{1}{2}\right)A\right),\\
\left\langle \Delta \hat{J}_y^2 \right\rangle &= \frac{J}{2}\left(1+\frac{1}{2}\left(J-\frac{1}{2}\right)\left(A+\sqrt{A^2+B^2}\cos(2\nu+2\delta)\right)\right),\\
\left\langle \Delta \hat{J}_z^2 \right\rangle &= \frac{J}{2}\left(1+\frac{1}{2}\left(J-\frac{1}{2}\right)\left(A-\sqrt{A^2+B^2}\cos(2\nu+2\delta)\right)\right),
\end{aligned}$$

where one defines $A = 1-\cos^{2J-2}(2\chi t)$,
$B = 4\sin(\chi t)\cos^{2J-2}(\chi t)$, and $\delta = \frac{1}{2}\arctan\left(\frac{B}{A}\right)$, the angle between the ellipse and the equator. Note in particular that, after squeezing, the average spin norm $\left\langle \hat{J}_x \right\rangle$ has decreased.

The $\hat{J}_x$ operator can be seen as describing the relative phase between two different spins along the $\mathbf{u}_z$ axis. By expanding the second order cosine in t, we obtain

$$\cos^{2J-1}(\chi t) = e^{(2J-1)\ln\cos(\chi t)} \approx e^{-(2J-1)(\chi t)^2/2}.$$

One can see that the average value of $\hat{J}_x$ of the state (1) is a gaussian decay, indicating that the phase blurs according to the following time scale

$$t_c \sim \frac{1}{|\chi|\sqrt{2J}}.$$

One can also observe the resurgence of phase coherence at the t_q times defined according to

$$t_q = \frac{q\pi}{|\chi|}, \quad q \in \mathbb{N},$$

when the cosine takes the values ± 1 in the evolution equation of the spin components and their variance.

II. How to squeeze?

At particular times $t_m = t_{1/2}$, the equation (1) can be rewritten as

$$|\psi(t_m)\rangle = \frac{e^{-i(\nu J + \frac{\pi}{4})}}{\sqrt{2}} \left|\frac{\pi}{2}, 0\right\rangle + \frac{e^{-i((\nu+\pi)J + \frac{\pi}{4})}}{\sqrt{2}} \left|-\frac{\pi}{2}, 0\right\rangle,$$

corresponding to a Schrödinger cat state in phase. For $\nu = \frac{\pi}{2} - \delta$, the term $\left\langle \Delta\hat{J}_y^2 \right\rangle$ is minimized and $\left\langle \Delta\hat{J}_z^2 \right\rangle$ maximized, and when $\nu = -\delta$, $\left\langle \Delta\hat{J}_z^2 \right\rangle$ is minimized while $\left\langle \Delta\hat{J}_y^2 \right\rangle$ is maximized. The increase ($+$ sign) and reduction of the variance ($-$ sign) are

$$V_\pm = \frac{J}{2}\left(\left(1 + \frac{1}{2}\left(J - \frac{1}{2}\right)A\right) \pm \frac{1}{2}\left(J - \frac{1}{2}\right)\sqrt{A^2 + B^2}\right).$$

For $J \gg 1$ and $|2\chi t| \ll 1$, the reduced variance V_- reaches its minimum

$$V_{\text{min}} \approx \frac{1}{2}\left(\frac{J}{3}\right)^{1/3},$$

at the following time

$$t_{\text{min}} = t_0 \approx \left(\frac{3}{8}\right)^{1/6} \frac{J^{-2/3}}{|\chi|}.$$

II. How to squeeze?

The normalized uncertainty product is

$$U_{yz} = \frac{4 \left\langle \Delta J_y^2 \right\rangle \left\langle \Delta J_z^2 \right\rangle}{| \left\langle J_x \right\rangle |^2},$$

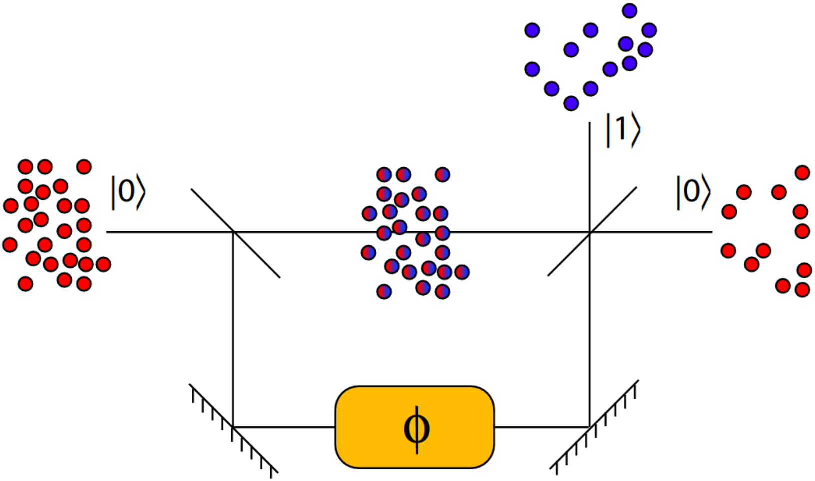
which can be calculated to

$$U_{yz} \approx 1 + \left(\frac{t}{t_0} \right)^6 .$$

Then the state remains in a state minimizing Heisenberg's inequality for $t < t_0$. The non-linear interaction will therefore tend to increase the product of uncertainty of the conjugated observables for long times, producing a state that does not saturate the Heisenberg inequality.

III.1. Case of a NOON state

One considers the case of a Ramsey interferometer, without loss of generality, the reasoning being generalizable to any atomic interferometer.



We then consider N atoms initially in the state $|0\rangle$ and after a $\pi/2$ pulse, each atom is then in the superposition of state $|0\rangle$ and $|1\rangle$.

$$|\Psi_i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right).$$

After the free evolution during an interrogation time T, a relative phase φ is accumulated between states

$$|\Psi_f\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\varphi}|1\rangle\right).$$

The probability $p(\varphi)$ that the final state Ψ_f is equal to the initial state Ψ_i makes it possible to evaluate φ according to

$$p(\varphi) = |\langle\Psi_i|\Psi_f\rangle|^2 = \cos^2\left(\frac{\varphi}{2}\right).$$

If we consider any observable $\hat{O}$ depending on a ζ parameter, measurements of $\hat{O}(\zeta)$ can traced back to ζ. The uncertainty in determining ζ for a single measurement is then

$$\delta\zeta = \frac{\Delta \hat{O}}{\left|\partial \left\langle \hat{O} \right\rangle / \partial\zeta\right|},$$

where $\left(\Delta \hat{O}\right)^2 \stackrel{\text{def.}}{=} \left\langle \hat{O}^2 \right\rangle - \left\langle \hat{O} \right\rangle^2$ is the variance of the observable $\hat{O}$. In the case considered, the uncertainty associated with the phase measurement is then

$$\Delta\varphi = \frac{\Delta p(\varphi)}{|\partial p(\varphi)/\partial\varphi|},$$

which in the case of a single achievement is $\Delta\varphi = \phi_0$. To improve this measure, the simplest way is to repeat the measure N times, in practice using a cloud of N atoms queried at the same time.

III.1. Case of a NOON state

If one performs N measurements $\{x_i\}$, then one obtains a good estimator of X with the mean value of x_i

$$X = \frac{1}{N} \sum_{i=1}^{N} x_i,$$

and an associated uncertainty, in the case of uncorrelated measurements

$$\Delta X = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\Delta x_i)^2} = \frac{\Delta x}{\sqrt{N}},$$

where the uncertainties of each measurement are assumed to be the same and equal to Δx.

Thus a phase measurement with a cloud of uncorrelated N atoms will give an error of

$$\Delta\varphi = \frac{\phi_0}{\sqrt{N}},$$

also known as *standard quantum limit*, by analogy with the corresponding optical case (Mach-Zehnder interferometer), or *quantum projection noise*. Cold atomic clocks have recently reached this limit.

We now consider the initial NOON-type state

$$|\Psi_i\rangle = \frac{1}{\sqrt{2}}\left(|N,0\rangle + |0,N\rangle\right),$$

which evolves, after interrogation, to the following state

$$|\Psi_f\rangle = \frac{1}{\sqrt{2}}\left(|N,0\rangle + e^{iN\varphi}|0,N\rangle\right).$$

III.1. Case of a NOON state

The probability $q(\varphi)$ that $|\Psi_f\rangle$ is equal to $|\Psi_i\rangle$ is then

$$q(\varphi) = \cos^2\left(N\frac{\varphi}{2}\right),$$

such that one may extract the value of the phase with an uncertainty given by

$$\Delta\varphi = \frac{\Delta q(\varphi)}{|\partial q(\varphi)/\partial\varphi|} = \frac{\phi_0}{N}.$$

The use of a maximally entangled state improves the accuracy of phase measurement by a factor of $\sqrt{N}$ over the standard quantum limit. The limit reached is then fundamental, and it is not possible to obtain a smaller error. This limit is then called **Heisenberg limit**.

Any two-level system is like an effective spin 1/2, possibly immersed in a homogeneous magnetic field. Without loss of generality, we will therefore consider a spin 1/2 in the following, and a Ramsey type interferometer.

During a measurement by Ramsey interferometry, one uses a set of N spin 1/2 initially in a given state noted $|0\rangle$. An initial $\pi/2$ pulse prepares the set of particles in the state

$$|\Psi_i\rangle = \left(\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\right)^{\otimes N},$$

which after a time of free evolution becomes

$$|\Psi_f\rangle = \left(\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\varphi}|1\rangle\right)\right)^{\otimes N}.$$

In order to measure the relative phase φ accumulated, a second $\pi/2$ pulse is applied to obtain a superposition of the form

$$|\Psi_m\rangle = (\alpha(\varphi)|0\rangle + \beta(\varphi)|1\rangle)^{\otimes N},$$

where $(\alpha, \beta) \in \mathbb{C}$ only depends on the relative phase φ. Excepted of the trivial cases where $\alpha = 1$ or 0, the measurement of the number of atoms in the state $|0\rangle$ is obtained with a statistical error given by the projection noise according to

$$\Delta N_0 = \sqrt{N|\alpha|^2\left(1 - |\alpha|^2\right)}.$$

To interpret this Ramsey sequence geometrically, one places oneself within the framework of the formalism of Bloch's sphere. Each particle is a spin $1/2$, $\hat{\mathbf{s}}_i$, the set forming a collective spin $\hat{\mathbf{J}}$ defined as the vectorial sum of the individual spins

$$\hat{\mathbf{J}} = \sum_{i=1}^{N} \hat{\mathbf{s}}_i.$$

This is the direction of the mean effective spin $\mathbf{u} = \left\langle \hat{\mathbf{J}} \right\rangle / \left| \left\langle \hat{\mathbf{J}} \right\rangle \right|$ which is represented on the Bloch sphere, and the associated uncertainty. Geometrically, a measurement of the populations of the two states corresponds to a projection on the $\mathbf{u}_z$ axis, while a $\pi/2$ pulse is a rotation of $\pi/2$ around the Oy axis. The relative phase φ corresponds to the azimuthal angle with respect to the direction Ox. In the case of a coherent state, the projection noise is the projection of the uncertainty circle on the Oz axis

One now considers a clock based on a Ramsey interferometer where the population measurement N_0 of the state $|0\rangle$ allows to estimate the phase $\varphi = (\omega - \omega_0)\,T$, where ω is the frequency of the wave coupling the two levels, ω_0 the Larmor frequency associated with the two-level system under consideration and T the interrogation time. One then obtains an rms error $\delta\omega$ on the measurement of the angular frequency of

$$\delta\omega = \frac{\Delta N_0}{\left|\partial \left\langle N_0 \right\rangle / \partial\omega\right|}.$$

By introducing the vector operators associated with the collective spin, we can rewrite this expression according to

$$\delta\omega = \frac{\Delta \hat{J}_z}{\left|\partial \left\langle \hat{J}_z \right\rangle / \partial\omega\right|}.$$

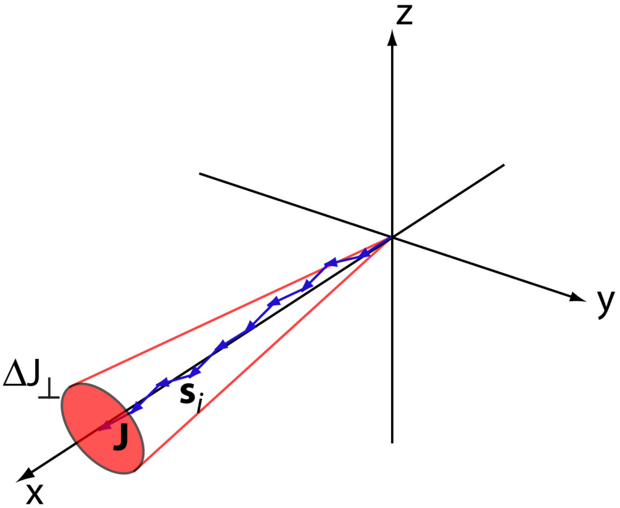
One now considers a clock based on a Ramsey interferometer where the population measurement N_0 of the state $|0\rangle$ allows to estimate the phase $\varphi = (\omega - \omega_0)\, T$, where ω is the frequency of the wave coupling the two levels, ω_0 the Larmor frequency associated with the two-level system under consideration and T the interrogation time. One then obtains an rms error $\delta\omega$ on the measurement of the angular frequency of

$$\delta\omega = \frac{\Delta N_0}{\left|\partial \langle N_0 \rangle / \partial\omega\right|}.$$

By introducing the vector operators associated with the collective spin, we can rewrite this expression according to

$$\delta\omega = \frac{\Delta \hat{J}_z}{\left|\partial \left\langle \hat{J}_z \right\rangle / \partial\omega\right|}.$$

The projection noise $\Delta \hat{J}_z$ has a direct impact on the accuracy of the frequency measurement. In the absolute, this variance $\Delta \hat{J}_z$ is not necessarily equal to $\Delta J_{\perp i}$, the variances of the two axes orthogonal to the direction of the mean spin $\mathbf{u}$. Nevertheless, in practice, a clock works at maximum sensitivity, ie a point of operation where $\left|\partial \left\langle \hat{J}_z \right\rangle / \partial \omega\right|$ is maximum. This point corresponds to a collective spin aligned with the *Oy* axis after interrogation. During the rotation of the second interrogation pulse, the average collective spin will therefore not be affected, only the fluctuations will eventually be modified during the rotation. In this case, the *Oz* axis is then orthogonal to $\mathbf{u}$ and thus the projection noise corresponds to a variance of a $\mathbf{v}$ component orthogonal to the direction of the mean collective spin.



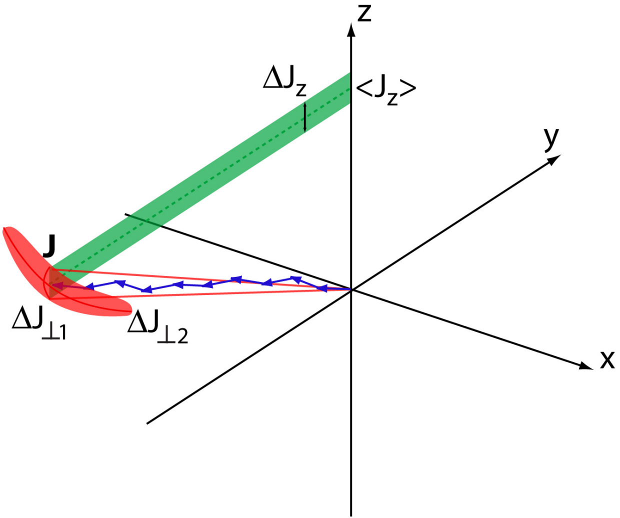
Geometric representation of a coherent state (CSS) in the Bloch sphere.

III.2. Squeezed states and quantum projection noise

In the case of a coherent state (CSS), this variance is independent of the **v** direction used, and is determined only in respect to the number of particles used. On the other hand, if one considers now the case of an SSS compressed spin state in the sense of the previous section, then there are two axes $\mathbf{v}_1$ and $\mathbf{v}_2$ of minimum and maximum variances respectively. If $\mathbf{v}_1$ is contained in the *Oyz* plane, then the variance ΔJ_z^2 will be reduced compared to the case of a coherent state. *The use of a compressed spin state reduces the statistical uncertainty due to projection noise in the case of a Ramsey interferometer*. This situation is geometrically illustrated in the formalism of the Bloch sphere.

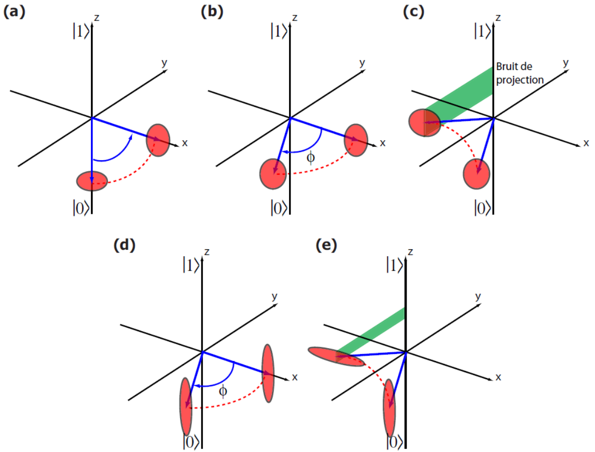
On the other hand, if $\mathbf{v}_2$ is contained in the *Oyz* plane, the variance ΔJ_z^2 will this time be higher than that of a coherent state, and thus the accuracy of the measurement will be degraded. In the other cases, we obtain a variance value between the two previous extremes.

III.2. Squeezed states and quantum projection noise



Geometric representation of a squeezed state (SSS) in the Bloch sphere.

III.2. Squeezed states and quantum projection noise



Ramsey interferometry with a coherent state (a-b-c) and a compressed state (d-e).

It should be noted that the reduction of measurement noise is not sufficient to increase the signal-to-noise ratio.

When a state is squeezed, if the angular dispersion of an orthogonal component decreases, the counterpart is that the orthogonal dispersion increases relative to a coherent state.

For a squeezed state, the mean value of the collective spin $\left\langle \hat{\mathbf{J}} \right\rangle$ will decrease as a norm compared to the case of a coherent state (where $\left\langle \hat{\mathbf{J}} \right\rangle \approx J = N/2$ it the limit $N \gg 1$).

This decrease in the mean collective spin norm translates in practice into a reduction in the contrast of the Ramsey fringes and thus a decrease in the useful signal.

If the state is too highly compressed, the signal reduction will outweigh the noise reduction and the signal-to-noise ratio will decrease. For a compressed state to be useful in the interferometric sense, it must be ensured that it retains sufficient coherence to provide a signal of sufficient amplitude.

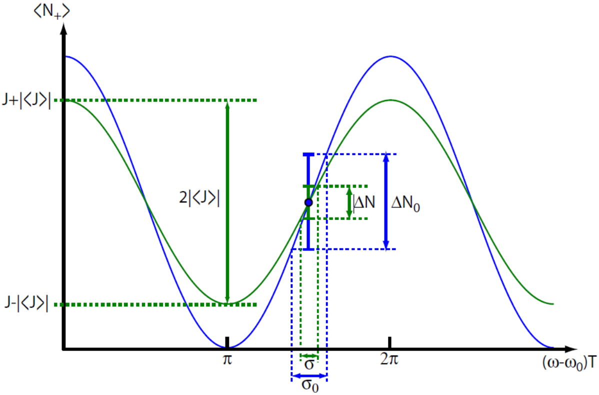


Illustration of projection noise reduction on a Ramsey interference signal.

III.2. Squeezed states and quantum projection noise

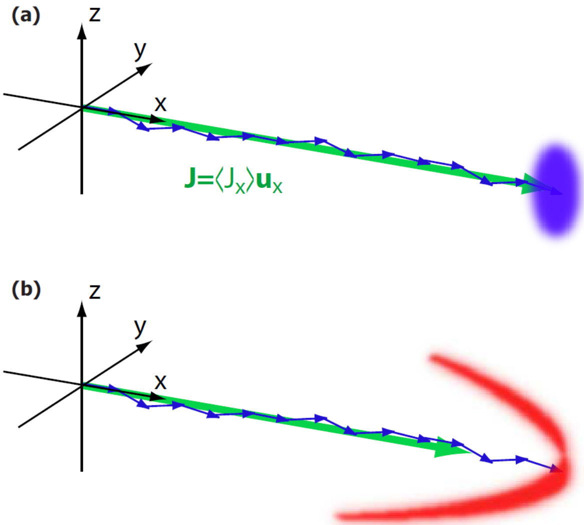


Illustration of the contrast reduction of a compressed state.

A state is squeezed of it exists a component $\hat{J}_\perp$ perpendicular to the direction of the mean collectif spin $\left\langle \hat{\mathbf{J}} \right\rangle$, and which variance is below the one of a coherent state (*i.e.* the standard quantum limit $J/2$). One defines then the spin squeezing factor ξ_S such that

$$\xi_S = \frac{\Delta J_\perp}{\sqrt{J/2}}.$$

In this approach, it is possible to rotate the collective spin to align it with Oz, *i.e.* $\left\langle \hat{\mathbf{J}} \right\rangle = \left\langle \hat{J}_z \right\rangle \mathbf{u}_z$ and such that $\Delta \hat{J}_\perp = \Delta \hat{J}_y$.

In the context of interferometers, the relevant observables of the problem are then the relative phase and the number of atoms in a state (or the relative population). In the collective spin formalism, the observable corresponding to the relative populations is $\hat{J}_z$. One will thus be interested only in the reduction of fluctuations of this observable and define the number squeezing factor ξ_N

$$\xi_N = \frac{\Delta \hat{J}_z}{\sqrt{J/2}},$$

i.e. it compares the fluctuations of the observable $\hat{J}_z$ to the standard quantum limit obtained for a coherent state aligned with the equator of Bloch's sphere. This criterion does not allow a complete estimation of the metrological gain obtained, as it does not include the decrease in contrast, which is due to the fact that $\left|\left\langle \hat{\mathbf{J}} \right\rangle\right| < J/2$ for highly squeezed states.

III.3. Squeezing factor for metrology

To quantify the gain obtained on the signal-to-noise ratio by using a compressed state instead of a coherent state, a quantization of this gain is introduced using the compression factor for Ramsey interferometry ξ_R

$$\xi_R^2 = \frac{N \Delta J_z^2}{\langle J_x \rangle^2 + \langle J_z \rangle^2}.$$

If we consider a phase measurement with a Ramsey interferometer, we get a statistical error on the measurement with a compressed state $\delta\varphi$ such that

$$\delta\varphi = \xi_R \delta\varphi_{\text{css}},$$

where $\delta\varphi_{\text{css}}$ is the statistical error obtained in the case of a consistent report. So we'll have a metrological gain if $\xi_R < 1$.